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Negative theorem of nearly monotone approximation in $L_p, o<p<1$

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Abstract

In [2] our direct theorem we see that , when we approximate an increasing function $f\ in\ L\rho[-1,1]$, we wish sometimes that the approximating polynomials be increasing also. However, this constraint, restricts very much the degree of approximation, that the polynomials can achieve, namely ; only the rate of ω_2^φ.

In[3] Kopotun, Leviatan and Prymak proved that relaxing the monotonicity requirement in intervals of measure zero near the end points allows the polynomials to achieve the rate of ω_3^φ.

On the other hand we show in this paper , that even when we relax the requirement of monotinicity of the polynomials on sets of measure approach zero, ω_4^φ is not reachable .

المستخلص

[1] النظرية المباشرة في التقريب الرتيب للدوال القياسية في الفضاء L_p عندما $0<p<1$ بدلالة مقياس النعومة ω_2^φ ،تبين انه من الممكن أن نقرب ألداله المتزايدة f بواسطة متعددة حدود أيضا متزايدة. إن قيد الرتابة في النتيجة أعلاه سيحدد درجة التقريب الأفضل بواسطة المتعددات الجبرية بدلالة المقياس ω_2^φ فقط ، ولتأكيد ذلك برهنا انه لا يمكن استبدال المقياس ω_2^φ للدالة f بالمقياس ω_3^φ،لذلك قدمنا النظرية النقيضة للنظرية المباشرة.

[2] العالم K.A. Kopotun برهن انه عندما يجعل متعددة الحدود الجبرية التقريبية تكون غير رتيبة مع دالة الهدف في بعض الفترات الجزئية من الفترة $[-1,1]$ التي تحتوي على النقاط الحرجة للدالة f بالإضافة إلى الفترات الجزئية التي تحتوي نهايات الفترة $[-1,1]$ والتي تكون أطوالها مقتربة إلى الصفر يمكن إن يحصل على التقريب بدلالة مقياس النعومة ω_3^φ.

برهنا انه لا يمكن استبدال مقياس النعومة ω_3^φ بالمقياس ω_4^φ حتى لو كانت المتعددة الجبرية غير رتيبة مع دالة الهدف في مجموعات جزئيه من الفترة $[-1,1]$ تقترب أطوالها إلى الصفر.

1. Introduction and Main result

Let $f \in L\rho, 0<\rho<1$ be increasing function on $I=[-1,1]$ *in* [2] we proved that there exists an increasing polynomials such that

$$\|f-\rho_n\|_{L\rho(I)} \le C(\rho)\omega_2^{\varphi}\left(f,\frac{1}{n}\right)_\rho , \qquad (1.1)$$

Where $C(\rho)$ is an absolute constant depending on ρ , and $\omega_2^{\varphi}(f;o)$ denote the Ditzain Totik modulus of smoothness of order two of f and ω_3^{φ} is the secand order Ditizin_Totik modulus of smoothness of f defined by

$$\omega_2^{\varphi}\left(f,\frac{1}{n}\right)_p = \sup_{0<h\le\delta}\left\|\Delta_{h\phi(.)}^{2}\left(f,\frac{1}{n}\right)\right\|_p , \delta \ge 0 , [1]$$

In [4] Devore proved that if $f \in C[-1,1]$ be non decreasing on $I=[-1,1]$ there exist non decreasing polynomials such that

$$\|f-\rho_n\|_{c(I)} \le C\,\omega_2\left(f,\frac{1}{n}\right)_\rho . \qquad (1.2)$$

However , even this improvement comes to ahalt; it can not be extended to w_4^{φ} ; and thus not to ω_k for any K > 3 .

In order to state our theorem we need some notation .

Given ε > 0 and a increasing function $f \in L\rho , 0<\rho<1$,We denote

$$E_n^{(1)}(f,\varepsilon)_\rho = \inf_{p_n \in \Pi_n \cap \Delta^1(I)} \|f-\rho_n\|_{L\rho(I)} .$$

Where the infimum is taken over all polynomials ρ_n of degree not exceeding n satisfying

$$meas\left(\{x:\rho'_n(x)\ge 0\}\cap I\right) \ge 2-\varepsilon$$

Then our main result in this paper is :

Theorem I: for each sequence $\bar{\varepsilon}=\{\varepsilon_n\}_{n=1}^{\infty}$, of non negative numbers tending to Zero , and each $0<\rho<1$, there exists an increasing function $f=f_{\overline{\in}} \in L_\rho(I)$ such that

$$\lim_{n\to\infty} \sup \frac{E_n^{(1)}(f;\varepsilon_n)_\rho}{\omega_4^{\varphi}\left(f,\frac{1}{n}\right)_\rho} = \infty \qquad (1.3)$$

2. A counter example (proof of theorem I)

In this article we have used C as an absolute constant which may differ on different occurrences , in this paper we will have to keep track of the constants , there fore we denote them by C_1, C_2 , …… ., We begin by recalling some simple properties of the Chebyshev polynomials for the interval [-2 , 2] , for v > 1 let $t_v(x)=\cos v \cos^{-1}\frac{x}{2}, x\in[-2,2]$.

Denote to the Chebyshev polynomial and let $Zj = 2\cos\frac{i\pi}{v}$, $j = 0,....,v$, be its extrema , In fact :

$$t_v(x) = \cos\left(v\cos^{-1}\frac{x}{2}\right)$$

$$t'_v\ (x) = -\sin\left(v\cos^{-1}\frac{x}{2}\right)\left(v\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\left(\frac{1}{2}\right)\right)$$

$$= \frac{v\sin\left(v\cos^{-1}\frac{x}{2}\right)}{2\ \sqrt{1-\left(\frac{x}{2}\right)^2}}$$

$$t'_v\ (x) = 0$$

$$v\sin\left(v\cos^{-1}\frac{x}{2}\right) = 0$$

$$\sin\left(v\cos^{-1}\frac{x}{2}\right) = 0$$

$$v\cos^{-1}\frac{x}{2} = \sin^{-1}0$$

$$v\cos^{-1}\frac{x}{2} = j\pi \quad ,j = 0,\ \ 1,\ 2,\ 3,......$$

$$\cos^{-1}\frac{x}{2} = \frac{j\pi}{v},\frac{x}{2} = \cos\frac{j\pi}{v}$$

$$Zj = 2\cos\frac{j\pi}{v} \quad ,j = 0,\ 1,\ 2,.......$$

Given $0 < b < \frac{1}{2}$, we take two points on both sides of $Zj, j = 1,......,v-1$, namely ,we set

$$Z_{j,l} = 2\cos\left(\frac{(j+b)\pi}{v}\right)\ and\ Z_{j,r} = 2\cos\left(\frac{(j-b)\pi}{v}\right),$$

$$\left|t_v\left(Z_{j,l}\right)\right| = \left|t_v\left(Z_{j,r}\right)\right| = \cos\pi\, b.$$

and

$$Z_{j,r} - Z_{j,l} = 2\cos\left(\frac{j\pi}{v} - \frac{b\pi}{v}\right) - 2\cos\left(\frac{j\pi}{v} + \frac{b\pi}{v}\right)$$

$$= 4\sin\frac{j\pi}{v}\sin\frac{b\pi}{v}$$

Then since $\sin u \le u$, $u \in [0,\pi]$, so that

$$Z_{j,r} - Z_{j,l} < 4\pi\frac{b}{v} \qquad (2.1)$$

We truncate the Chebyshev polynomial by setting

$$t_v^*(x) = t_{v,b}^*(x) = \begin{cases} \cos\pi b & t_v(x) > \cos\pi b \\ -\cos\pi b & t_v(x) < -\cos\pi b \\ t_v(x) & O.W \end{cases}$$

$$\sin ce \qquad 1 - \cos\pi b = 2\sin^2\frac{b\pi}{2} < 5b^2 \qquad (2.2)$$

For any $x \in I$,it follows by the monotonicity of the areas as we go away from the origin , and the alternation in sign of these areas ,that

$$\left|\int_{Z_{\left[\frac{v}{2}\right],l}}^{Z_{\left[\frac{v}{2}\right],r}} \left(t_v(u) - t_v^*(u)\right)du\right| \le \int_{Z_{\left[\frac{v}{2}\right],l}}^{Z_{\left[\frac{v}{2}\right],r}} \left|t_v(u) - t_v^*(u)\right|du$$

$$\le \left|Z_{\left[\frac{v}{2}\right],r} - Z_{\left[\frac{v}{2}\right],l}\right| \sup\left|t_v(u) - t_v^*(u)\right|$$

$$\le 4\pi\frac{b}{v}\left(t_v(u) - t_v^*(u)\right)$$

$$\le 4\pi\frac{b}{v}\left(1 - \cos\pi b\right)$$

$$< 4\pi\frac{b}{v}5b^2$$

$$= c_1\frac{b^3}{v}, \qquad (2.3)$$

$$Now \quad \sin ce\,[0,x] \subset \left[Z_{\left[\frac{v}{2}\right],l}, Z_{\left[\frac{v}{2}\right],r}\right] \subset I = [-1,1]$$

Then

$$\left|\int_0^x \left(t_v(u) - t_v^*(u)\right)du\right| \le \left|\int_{Z_{\left[\frac{v}{2}\right],l}}^{Z_{\left[\frac{v}{2}\right],r}} \left(t_v(u) - t_v^*(u)\right)du\right| < 4\pi\frac{b}{v}5b^2 = c_1\frac{b^3}{v} \qquad (2.4)$$

Where we applied (2.1)

Now , given $n \ge 1$ *and* $0 < b < \frac{1}{2}$ *,let* $v = \left[b_n^{\frac{3}{4}} \right] + 2$, *where* $[a]$

denotes the largest integer not exceeding a .put

$t_{v,b} = t_v + \cos \pi b$ *,and* $t_v^{\sim} = t_{v,b}^{*} + \cos \pi b$.

Finally

$$T_{v,b}(x) = \int_0^x t_{v,b}(u)du \quad and \quad f_{n,b}(x) = \int_0^x t_{v,b}^{\sim}(u)du \quad , x \in I$$

let $x_1 < x_2$

$$f_{n,b}(x_1) = \int_0^{x_1} t_{v,b}(u)du < \int_0^{x_2} t_{v,b}(u)du = f_{n,b}(x_2).$$

Obviously $f_{n,b}$ is a increasing function on I and it readily by (2.4) that

$$\|f_{n,b} - T_{v,b}\|_p = \left(\int_{-1}^{1} \left| \int_0^x t_{v,b}^{\sim}(u)du - \int_0^x t_{v,b}(u)du \right|^p du \right)^{\frac{1}{p}}$$

Now

$$\left| f_{n,b} - T_{v,b} \right| = \left| \int_0^x \left(t_{v,b}^{\sim}(u) - t_{v,b}(u) \right) du \right|$$

$$= \left| \int_0^x \left(t_{v,b}^{*}(u) + \cos \pi b - t_{v,b}(u) - \cos \pi b \right) du \right|$$

$$= \left| \int_0^x \left(t_{v,b}^{*}(u) - t_{v,b}(u) \right) du \right|$$

$$\le C_1 \frac{b^3}{v} = \frac{C_1 b^3}{\left[b_n^{\frac{3}{4}} \right] + 2} \le C_1 \frac{b^3}{b_n^{\frac{3}{4}}} = C_1 \frac{b^{\frac{9}{4}}}{n},$$

And

$$\|f_{n,b} - T_{v,b}\|_p \le \left(\int_{-1}^{1} \left| C_1 \frac{b^{\frac{9}{4}}}{n} \right|^p du \right)^{\frac{1}{p}}$$

$$\le C_1 \frac{b^{\frac{9}{4}}}{n} \left(\int_{-1}^{1} du \right)^{\frac{1}{p}} \le C_1 2^{\frac{1}{p}} \frac{b^{\frac{9}{4}}}{n} \ ,$$

Then

$$\left\| f_{n,b} - T_{v,b} \right\|_p \le C_1 2^{\frac{1}{p}} \frac{b^3}{v} = C_1 2^{\frac{1}{p}} \frac{b^3}{\left[b^{\frac{3}{4}} n \right] + 2} \le C_1 2^{\frac{1}{p}} \frac{b^{\frac{9}{4}}}{n} \qquad (2.5)$$

Where we denote by $\|\cdot\|_{Lp(J)}$ the quas – norm taken on the interval J , and when the norm is on J . We suppress the subscript ,

If we set $\tilde{Z}_{j,L} = 2\cos\left(\frac{\left(j+\frac{b}{2}\right)\pi}{v}\right)$, and $\tilde{Z}_{j,r} = 2\cos\left(\frac{\left(j-\frac{b}{2}\right)\pi}{v}\right)$,

Then we have for all j for which $Z_j \in I$

$$\begin{aligned}
\tilde{Z}_{j,r} - Z_j &= 2\cos\left(\frac{\left(j-\frac{b}{2}\right)\pi}{v}\right) - 2\cos\frac{j\pi}{v} \\
&= 2\cos\left(\frac{j\pi}{v} - \frac{b\pi}{2v}\right) - 2\cos\frac{j\pi}{v} \\
&= 2\cos\left(\frac{j\pi}{v} - \frac{b\pi}{4v} - \frac{b\pi}{4v}\right) - 2\cos\left(\frac{j\pi}{v} - \frac{b\pi}{4v} + \frac{b\pi}{4v}\right) \\
&= 2\cos\left(\frac{\left(j-\frac{b}{4}\right)\pi}{v} - \frac{b\pi}{4v}\right) - 2\cos\left(\frac{\left(j-\frac{b}{4}\right)\pi}{v} + \frac{b\pi}{4v}\right) \\
&= 4\sin\left(\frac{\left(j-\frac{b}{4}\right)\pi}{v}\right)\sin\frac{b\pi}{4v}
\end{aligned}$$

Then since $(\sin b\pi/4 > 3b/4)$ for b satisfying

$b\pi/4 < \pi/6 \ \ if \ \ 0 < b < \frac{1}{2} \ \ and \ \ \sin u \ge \frac{2}{\pi} u, \ \ u \in \left[0, \frac{\pi}{2}\right]$

$$\begin{aligned}
\tilde{Z}_{j,r} - Z_j &= 4\sin\frac{\left(j-\frac{b}{4}\right)\pi}{v}\sin\frac{b\pi}{4v} \\
&\ge 4\frac{2}{\pi}\left(\frac{\left(j-\frac{b}{4}\right)\pi}{v}\right)\frac{3b}{4} = \frac{6\left(j-\frac{b}{4}\right)b}{v} > \frac{b}{v} \qquad (2.6)
\end{aligned}$$

And

$$Z_j - \tilde{Z}_{j,l} = 2\cos\frac{j\pi}{v} - 2\cos\left(\frac{\left(j+\frac{b}{2}\right)\pi}{v}\right)$$

$$= 2\cos\left(\frac{j\pi}{v} + \frac{b\pi}{4v} - \frac{b\pi}{4v}\right) - 2\cos\left(\frac{j\pi}{v} + \frac{b\pi}{4v} + \frac{b\pi}{4v}\right)$$

$$= 2\cos\left(\frac{\left(j+\frac{b}{4}\right)\pi}{v} - \frac{b\pi}{4v}\right) - 2\cos\left(\frac{\left(j+\frac{b}{4}\right)\pi}{v} + \frac{b\pi}{4v}\right)$$

$$= 4\sin\frac{\left(j+\frac{b}{4}\right)\pi}{v}\sin\frac{b\pi}{4v}$$

$$\geq 4\frac{2}{\pi}\left(\frac{\left(j+\frac{b}{4}\right)\pi}{v}\frac{3b}{4} = \frac{6\left(j+\frac{b}{4}\right)b}{v}\right) > \frac{b}{v} \qquad (4.2.6')$$

Let j be odd and since $\sin b\pi/4 > 3b/4$

For b satisfying $b\pi/4 < \pi/6$, we have

$T'_{v,b}(x) = t_{v,b}(x) \leq -\cos b\pi/2 + \cos b\pi \;, x \in \left[\tilde{Z}_{j,l}, \tilde{Z}_{j,r}\right].$

For example

$$t_{v,b}(x) = t_v(x) + \cos\pi b \leq t_v\left(\tilde{Z}_{j,l}\right) + \cos\pi b = \cos v \cos^{-1}\frac{2\cos\left(\frac{j\pi}{v} + \frac{b\pi}{2v}\right)}{2} + \cos\pi b$$

$$= \cos v\left(\frac{j\pi}{v} + \frac{b\pi}{2v}\right) + \cos\pi b$$

$$= \cos\left(j\pi + \frac{b\pi}{2}\right) + \cos\pi b$$

$$= -\cos\frac{b\pi}{2} + \cos\pi b$$

And

$-\cos\pi b/2 + \cos\pi b = -\sin b\pi/4 \sin 3b\pi/4$

In fact,

$$-\cos \pi b/2+\cos \pi b=-\cos\left(\frac{b\pi}{4}-\frac{3b\pi}{4}\right)+\cos\left(\frac{b\pi}{4}+\frac{3b\pi}{4}\right)$$

$$=-2\sin\frac{b\pi}{4}\sin\frac{3b\pi}{4}$$

$$<-2\frac{3b}{4}\,\frac{3b}{2}=-\frac{9b^2}{4} \quad ,x\in\left[\tilde{Z}_{j,l},\tilde{Z}_{j,r}\right]$$

Then

$$T'_{v,b}(x)=t_{v,b}(x)\le-\cos b\pi/2+\cos b\pi=-2\sin b\pi/4\sin 3b\pi/4$$

$$<-2\frac{3b}{4}\,\frac{3b}{2}=-\frac{9b^2}{4} \quad ,x\in\left[\tilde{Z}_{j,l},\tilde{Z}_{j,r}\right], \tag{2.7}$$

Then since $I\subset[-2,2]$, it follows by the Bernstein inequality

$$\left(\left\|p'_n\right\|_{L_{P[-2,2]}}\le\frac{cn}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\left\|p_n\right\|_{Lp[-2,2]}\right) \text{ that}$$

$$\left\|T_{v,b}^{(4)}\right\|_{lp[-2,2]}=\left\|t_{v,b}^{(3)}\right\|_{L_{lp[-2,2]}}=\left\|t_v^{(3)}\right\|_{Lp[-2,2]}\le\frac{C_2\;V^3}{\left(1-\left(\frac{1}{2}\right)^2\right)^{\frac{3}{2}}}\left\|t_v\right\|_{Lp[-2,2]}=C_3\,v^3,$$

In fact

$$\left\|T_{v,b}^{(4)}\right\|_{Lp[-2,2]}=\left\|\frac{d}{dx}\left(\int_0^x t_{v,b}^{(3)}(u)\,du\right)\right\|_{Lp[-2,2]}=\left\|t_{v,b}^{(3)}\right\|_{Lp[-2,2]}$$

$$=\left\|t_v^{(3)}(u)\right\|_{Lp[-2,2]}\le C_2\;V^3\left\|\frac{1}{\left(\sqrt{1-\left(\frac{x}{2}\right)^2}\right)^3}t_v\right\|_{Lp[-2,2]}$$

$$\le\frac{C_2\,V^3}{\left(\sqrt{1-\left(\frac{1}{2}\right)^2}\right)^3}\left\|t_v\right\|_{Lp[-2,2]}$$

Hence by (2.5)

$$=\frac{C_2\,V^3}{\left(\sqrt{1-\left(\frac{1}{2}\right)^2}\right)^3}\left(\int_{-2}^{2}\left|\cos v\cos^{-1}\frac{x}{2}\right|^p dx\right)^{\frac{1}{p}}$$

$$\le C_3\,V^3\,4^{\frac{1}{P}}$$

$$
\begin{aligned}
\omega_4^{\varphi}\left(f_{n,b},\frac{1}{n}\right)_p &= \omega_4^{\varphi}\left(f_{n,b}-T_{v,b}+T_{v,b},\frac{1}{n}\right)_p \\
&\le \omega_4^{\varphi}\left(f_{n,b}-T_{v,b},\frac{1}{n}\right)_p+\omega_4^{\varphi}\left(T_{v,b},\frac{1}{n}\right)_p \\
&\le C(p)\left\|f_{n,b}-T_{v,b}\right\|_p+\frac{C(p)}{n^4}\left\|T_{v,b}^{(4)}\right\|_p \\
&\le C(p)\frac{b^{\frac{9}{4}}}{n}+\frac{C(p)v^3}{n^4} \\
&= C(p)\frac{b^{\frac{9}{4}}}{n}+C(p)\frac{\left(\left[b^{\frac{3}{4}}n\right]+2\right)^3}{n^4} \\
&\le C(p)\frac{b^{\frac{9}{4}}}{n}+C(p)\frac{\left(b^{\frac{3}{4}}n\right)^3}{n^4}
\end{aligned}
$$

$$
\begin{aligned}
&= C(p)\frac{b^{\frac{9}{4}}}{n}+C(p)\frac{b^{\frac{9}{4}}n^3}{n^4} \\
&= C(p)\frac{b^{\frac{9}{4}}}{n}+C(p)\frac{b^{\frac{9}{4}}}{n} \\
&= C(p)\frac{b^{\frac{9}{4}}}{n}=C_1(p)\frac{b^{\frac{9}{4}}}{n}\,, \qquad (2.8)
\end{aligned}
$$

Next we need a simple lemma

Lemma (2.9) there exists a constant C_4 such that
For any interval $J\subseteq I$, We have the following .
For any measurable sets $E\subseteq I$, if

$$p_n'(x)\ge 0,\ x\in J\setminus E, \qquad (2.10)$$

Then

$$\left\|f_{n,b}-p_n\right\|_{L_{p(J)}}>\frac{b^2|J|}{n}-\frac{C_4}{n}\left(b^{\frac{9}{4}}+b|E|+\frac{b^{\frac{5}{4}}}{n}\right). \qquad (2.11)$$

Proof : let J_o denote the middle third of J . We consider two cases . First we assume that J_o contains at most one of the $Z_j's$.Then by the definition of v we get

$$|J|<C_5\ \frac{1}{v}<C_5\ \frac{b^{\frac{-3}{4}}}{n} \tag{2.12}$$

and

$$\frac{b^2\,|J|}{n}<C_5\ \frac{b^2\,b^{\frac{-3}{4}}}{n^2}=C_5\ \frac{b^{\frac{5}{4}}}{n^2}$$

$$\frac{b^2\,|J|}{n}-C_5\ \frac{b^{\frac{5}{4}}}{n^2}<0$$

Then by (2.11) we have

$$\|f_{n,b}-p_n\|_{lp(J)}\geq 0>\frac{b^2\,|J|}{n}-C_5\ \frac{b^{\frac{5}{4}}}{n^2}\ , \tag{2.13}$$

On the other hand , if J_o contains at least two extremes of Z_j , then it contains at least $2C_6\,v|J|$ extrema , for some constant C_6 ,these extremes satisfy (4.2.6) and $(4.2.6^{/})$ and about half of them (and at least one) have odd indices , then together with (4.2.6) we conclude that

$$meas\ \left(J_o\quad \left\{x: T'_{v,b}(x)<-\frac{9b^2}{4}\right\}\right)\geq\frac{1}{2}\frac{2b}{v}C_6\,v|J|=C_6\,b|J|\ , \tag{2.14 }$$

Now ,if $C_6 b|J|\leq|E|$, then

$$\left\|f_{\ n,b}-p_n\right\|_{L_{p(J)}}\geq 0>\frac{b^2|J|}{n}-\frac{b|E|}{n\,C_6}, \tag{2.15}$$

Other wise , $C_6\,b|J|>|E|$.

Then by (4.2.14) there is a point $x_o\in J_o\setminus E$, for which

$$T'_{v,b}(x_o)<-\frac{9b^2}{4}$$

Hence ,(4.2.10) yields ,

$$\frac{9b^2}{4}\leq -T'_{v,b}(x_o)<p'_n\ (x_o)-T'_{v,b}(x_o)\leq\frac{2}{|J|^{\frac{1}{p}}}\frac{n}{\sqrt{1-\left(\frac{1}{3}\right)^2}}\|p_n-T_{v,b}\|_{Lp(J)}$$

$$\leq\frac{2}{|J|}\frac{n}{\sqrt{1-\left(\frac{1}{3}\right)^2}}\|p_n-T_{v,b}\|_{Lp(J)}$$

Where we used the Bernstein inequality

Therefore from (4.2.6)

$$\frac{9b^2}{4} \le \frac{2n}{|J|\sqrt{1-\left(\frac{1}{3}\right)^2}} \|P_n - T_{v,b}\|_{Lp(J)} = \frac{2n}{|J|\frac{2\sqrt{2}}{3}} \|p_n - T_{v,b}\|_{Lp(J)}$$

$$\frac{|J|\sqrt{2}}{3n} \cdot \frac{9b^2}{4} \le \|p_n - T_{v,b}\|_{Lp(J)}$$

$$\frac{3\sqrt{2}}{4}\frac{b^2|J|}{n} \le \|p_n - T_{v,b}\|_{Lp(J)}$$

Now

$$\frac{b^2|J|}{n} \le \frac{3\sqrt{2}}{4}\frac{b^2|J|}{n} \le \|p_n - T_{v,b}\|_{Lp(J)}$$
$$\le \|P_n - f_{n,b}\|_{Lp(J)} + \|f_{n,b} - T_{V,b}\|_{LP(J)}$$
$$\le \|P_n - f_{n,b}\|_{Lp(J)} + C_1(p)\frac{b^{\frac{9}{4}}}{n}.$$

Then

$$\|f_{n,b} - p_n\|_{Lp(J)} \ge \frac{b^2|J|}{n} - C_1(p)\frac{b^{\frac{9}{4}}}{n}, \qquad (2.16)$$

Taking $C_4 = \max\left\{C_5, \frac{1}{C_6}, C_1(\rho)\right\}$, (2.11) now follows by combining $(2.13), (2.15) \, and \, (2.16)$ ♣

We are now in a position to define $f_{\bar{\varepsilon}} = f$, for a given sequence $\bar{\varepsilon} = \{\varepsilon_n\}$.

Let $b_n = \left(\max\left\{\varepsilon_n^2, \frac{1}{n}\right\}\right)^{\frac{2}{5}}$, and set $d_0 = 1$ and

$$d_j = \frac{bn_j^{\frac{9}{4}}}{n_j} d_{j-1} = \prod_{v=1}^{j} \frac{bn_v^{\frac{9}{4}}}{n_v}, \, j \ge 1,$$

Where the sequence $\{n_v\}$ is defined by induction as follows .First ,we choose n_1 so large that $b_{n_1}^{\frac{1}{8}} < \frac{1}{12}$ (as needed in (2.18) below) and $J_o = I$. suppose that $\{n_1, n_2, \ldots\ldots\ldots, n_{\sigma-1}\}$ and $J_{\sigma-2} \subseteq J_{\sigma-3} \subseteq \ldots\ldots\ldots \subseteq J_0, \sigma \ge 2$, have been defined

Then put

$$F_{\sigma-1} = \sum_{j=1}^{\sigma-1} d_{j-1} f_{nj}, b_{nj},$$

And let $J_{\sigma-1}$ be an interval such that $J_{\sigma-1} \subseteq J_{\sigma-2}$ and

$$F'_{\sigma-1}(x) = 0, x \in J_{\sigma-1} \qquad (2.17)$$

Let $N_{1.\sigma}$ be such that

$$\left|J_{\sigma-1}\right| \geq b_n^{\frac{1}{8}}, n \geq N_{1,\sigma} \tag{2.18}$$

And let

$$N_{2,\sigma} = \left(\frac{\left\| F_{\sigma-1}^{(2)} \right\|_{L\rho\left(J_{\sigma-1}\right)}}{d_{\sigma-1}} \right)^{10} \tag{2.19}$$

Finally ,we take

$$n_\sigma > \max\left\{n_{\sigma-1}, N_{1,\sigma}, N_{2,\sigma}\right\}$$

So big that the function $f'_{n_\sigma, b_{n_\sigma}}$ Oscillates a few times inside the interval $(J_{\sigma-1})$ and since it vanishes on some interval in each Oscillation , that is , inside $J_{\sigma-1}$, there exists an interval $J_\sigma \subset J_{\sigma-1}$ as required in (2.17)

Now denote

$$\Phi_\sigma = \sum_{j=\sigma}^{\infty} d_{j-1} f_{nj}, _{bnj} ,$$

Where the convergence of the series is justified by the definition of the d_j's and

$$\left|f_{n, b_n}\right| = \left|\int_0^x t_{v,bn}(u)du\right|$$

$$\leq \int_0^x \left|t_{v,b_n}(u)du\right|$$

$$= \int_o^x \left|t_v(u) + \cos\pi b\right| du$$

$$\leq 2\int_0^x du \leq 2\int_0^1 du = 2$$

Now

$$\left\|\Phi_\sigma\right\|_{l\,p(J_o)} \leq 8d_{\sigma-1} \tag{2.20}$$

$$\left\|\Phi_\sigma\right\|^p{}_{L\rho(J_o)} = \left\|\sum_{j=\sigma}^{\infty} d_{j-1} f_{nj}, _{bnj}\right\|_{L\rho(J_o)}$$

$$\leq 2\sup\left|\sum_{j=\sigma}^{\infty} d_{j-1} f_{nj}, _{bnj}\right|$$

In fact $\leq 2\sup\sum_{j=\sigma}^{\infty}\left|d_{j-1}\right|\left|f_{nj}, _{bnj}\right|$

$$\leq \left(d_{\sigma-1}\left(1 + \frac{bn_\sigma^{9/4}}{n_\sigma} + \frac{bn_\sigma^{9/4}}{n_\sigma}\frac{bn_{\sigma+1}^{9/4}}{n_{\sigma+1}} + \ldots\ldots\ldots\right)\right)(2^2)$$

$$= d_{\sigma-1}\sum_{j=0}^{\infty} 2^{-j}\left(2^{2}\right)$$
$$= 8d_{\sigma-1}$$

So we define

$$f = f_{\bar{\varepsilon}} = \sum_{j=1}^{\infty} d_{j-1} f_{nj,bnj}$$

And we prove

Lemma(2.21) For each $\sigma \geq 1$ we have

$$\omega_4^{\varphi}\left(f, \frac{1}{n_\sigma}\right)_\rho \leq C_2(\rho) d_\sigma \qquad (2.22)$$

Proof : First , by (2.20)

$$\omega_4^{\varphi}\left(\Phi_{\sigma+1}, \frac{1}{n_\sigma}\right)_\rho \leq C_3(\rho)\left\|\Phi_{\sigma+1}\right\|_{Lp(J_0)} \leq C_3(\rho) d_\sigma \qquad (2.23)$$

At the same time , (4.2.8) yields

$$\omega_4^{\varphi}\left(d_{\sigma-1} f_{n\sigma, b_{n_\sigma}}, \frac{1}{n_\sigma}\right)_\rho \leq d_{\sigma-1} C_4 \frac{bn_\sigma^{9/4}}{n_\sigma} = C_4 d_\sigma \qquad (2.24)$$

Finally

$$\omega_4^{\varphi}\left(F_{\sigma-1}, \frac{1}{n_\sigma}\right)_\rho \leq C_4(\rho)\omega_2^{\varphi}\left(F_{\sigma-1}, \frac{1}{n_\sigma}\right)_\rho$$
$$\leq \frac{C_5(\rho)}{n_\sigma^2}\left\|F_{\sigma-1}^{(2)}\right\|_{L\rho(J_0)}$$
$$= C_5(\rho)\frac{\left\|F_{\sigma-1}^{(2)}\right\|_{L\rho(J_0)}}{d_{\sigma-1}} n_\sigma^{-1/10}\left(\frac{1}{n_\sigma^{2/5} \quad bn_\sigma}\right) d_\sigma^{9/4}$$
$$= C_5(\rho) N_{2,\sigma}^{\frac{1}{10}} n_\sigma^{\frac{-1}{10}}\left(\frac{1}{n_\sigma^{2/5} \quad bn_\sigma}\right)^{9/4} d_\sigma$$
$$< C_5(\rho) n_\sigma^{\frac{1}{10}} n_\sigma^{\frac{-1}{10}}\left(\frac{1}{n_\sigma^{2/5} \quad bn_\sigma}\right)^{9/4} d_\sigma$$
$$\leq C_5(\rho)\left(\frac{1}{\frac{1}{b_{n\sigma}} \quad bn_\sigma}\right)^{9/4} d_\sigma \qquad (2.25)$$
$$= C_5(\rho) d_\sigma$$

By virtue of (2.19) and the definition of $d_{n\sigma}, d_\sigma$ and n_σ .

$$\omega_4^{\varphi}\left(f, \frac{1}{n_\sigma}\right)_\rho \leq \omega_4^{\varphi}\left(\Phi_{\sigma+1}, \frac{1}{n_\sigma}\right)_\rho + \omega_4^{\varphi}\left(d_{\sigma-1} f_{n\sigma, b_{n_\sigma}}, \frac{1}{n_\sigma}\right)_\rho + \omega_4^{\varphi}\left(F_{\sigma-1}, \frac{1}{n_\sigma}\right)_\rho$$
$$\leq C_3(\rho) d_\sigma + C_4 d_\sigma + C_5(\rho) d_\sigma$$

Now Then Lemma (2.21) follows by combining (2.23) , (2.24) and (2.25) ♣
The last Lemma that we need is

Lemma (2.26) There is an absolute constant C_7 such that whenever $E \subset I$ is a measurable set satisfying

$$|E| \le \varepsilon_{n\sigma} , \tag{2.27}$$

And ρ_{n_σ} is a polynomial satisfying

$$\rho'_{n_\sigma}(x) \ge 0, x \in I \setminus E, \tag{2.28}$$

Then

$$\left\| f - \rho_{n_\sigma} \right\|_{L\rho(I\setminus E)} \ge \left(b_{n_\sigma}^{-1/8} - C_7 \right) d_\sigma , \tag{2.29}$$

Proof : by (2.14) we have $F_{\sigma-1}$ is constant on $J_{\sigma-1}$, we may write

$$f(x) = d_{\sigma-1} \; f_{n_\sigma , b n_\sigma}(x) + \Phi_{\sigma+1}(x) + M \;, x \in J_{\sigma-1} \tag{2.30}$$

Let $Q_{n_\sigma} = \frac{1}{d_{\sigma-1}} \left(\rho_{n_\sigma} - M \right)$

Then it follows from (2.28)

$$Q'_{n_\sigma}(x) \ge 0 , x \in J_{\sigma-1} \setminus E$$

Thus by virtue of Lemma (2.9)

$$\left\| Q_{n\sigma} - f_{n_\sigma , bn_\sigma} \right\|_{L\rho(J_{\sigma-1})} \ge \frac{b_{n_\sigma}^2 |J_{\sigma-1}|}{n_\sigma} - \frac{C_4}{n_\sigma} \left(\left(b^{9/4}_{n\sigma} + b_{n\sigma} |E| \right) + \frac{b_{n\sigma}^{5/4}}{n_\sigma} \right)$$

The definition of n_6 and (2.17)

$$b_{n_\sigma}^2 \left| J_{\sigma-1} \right| = b_{n_\sigma}^{17/8} \left(\frac{\left| J_{\sigma-1} \right|}{b_{n_\sigma}^{1/8}} \right) \ge b_{n_\sigma}^{17/8}$$

On the other hand , (2.24) and the definition

Of b_{n_σ} imply $\varepsilon_{n_\sigma} \le b_{n_\sigma}^{5/4}$

and

$$b_{n_\sigma} |E| \le b_{n_\sigma} \varepsilon_{n_\sigma} \le b_{n_\sigma}^{9/4}$$

Since $b_{n_\sigma} \ge \left(\frac{1}{n_\sigma} \right)^{2/5}$

Then

$$\frac{b_{n_\sigma}^{5/4}}{n_\sigma} \le b_{n\sigma}^{15/4} < b_{n\sigma}^{9/4}$$

Hence (2.28) implies

$$\left\| Q_{n\sigma} - f_{n\sigma}, b_{n\sigma} \right\|_{L\rho(J_{\sigma-1})} \ge \frac{1}{n_\sigma}\left(b_{n\sigma}^{17/8} - 3C_4 b_{n\sigma}^{9/4} \right) = \frac{b_{n\sigma}^{9/4}}{n_\sigma}\left(b_{n\sigma}^{-1/8} - 3C_4 \right)$$ In other words .

$$\left\| \rho_{n_\sigma} - M - d_{\sigma-1} f_{n_\sigma}, b_{n_\sigma} \right\|_{L\rho(J_{\sigma-1})} \ge d_{\sigma-1} \frac{b_{n_\sigma}^{9/4}}{n_\sigma}\left(b_{n_\sigma}^{-1/8} - 3C_4 \right)$$

$$= d_\sigma \left(b_{n_\sigma}^{-1/8} - 3C_4 \right),$$

In view of (2.30) , follows from (2.20) that

$$\left\| f - \rho_{n_\sigma} \right\|_{L\rho(I)} \ge \left\| f - \rho_{n_\sigma} \right\|_{L\rho(J_{\sigma-1})} \ge \left\| \rho_{n_\sigma} - M - d_{\sigma-1} f_{n_\sigma}, b_{n_\sigma} \right\|_{L\rho(J_{\sigma-1})} - \left\| \Phi_{\sigma+1} \right\|_{L\rho(J_{\sigma-1})}$$

$$\ge \left(\left(b_{n_\sigma}^{-1/8} - (3C_4 + 8) \right) d_\sigma \right),$$

And Lemma (2.26) is proved with $C_7 = 3C_4 + 8$

The proof of (.1.3) now follows from Lemmas (2.21)and(2.26) since

$$\lim_{n\to\infty} \sup \rho \frac{E_n^{(1)}(f, \varepsilon_n)_\rho}{\omega_4^\varphi\left(f, \frac{1}{n}\right)_\rho} \ge \lim_{n_\sigma\to\infty} \sup \rho \frac{E_{n\sigma}^{(1)}\left(f, \varepsilon_{n_\sigma}\right)_\rho}{\omega_4^\varphi\left(f, \frac{1}{n_\sigma}\right)_\rho} \ge \lim_{n_\sigma\to\infty} \sup \rho \frac{\left\| f - \rho_{n_\sigma} \right\|_{L\rho}}{C_2(\rho) d_\sigma}$$

$$\ge \lim_{n_\sigma\to\infty} \frac{1}{C_2(\rho)}\left(b_{n_\sigma}^{-1/8} - C_7 \right) = \infty$$

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